1. **TEXT AND OTHER MATERIALS:**
   - **Check Learning Resources** in **shared class files**
   - Calculus Wiki-book: https://en.wikibooks.org/wiki/Calculus (Main Reference e-Book)
   - Paul's Online Math Notes: http://tutorial.math.lamar.edu
   - Calculus. Early Transcendental Functions by Larson & Edwards, **5th** Editions
   - Calculus. Early Transcendental Functions by Larson & Edwards, **6th** Editions
   - http://ocw.mit.edu/resources/res-18-001-calculus-online-textbook-spring-2005/textbook/

2. Tutorial: http://archives.math.utk.edu/visual.calculus/
   Tutorial, Animation: http://www2.latech.edu/~schroder/animations.htm
   Tutorial: https://www.math.ucdavis.edu/~kouba/ProblemsList.html
   Tutorial: http://www.straighterline.com/landing/online-calculus-video-tutorials/#.Vb_en_lVhBc

3. **Technology Resources**:
   - Desmos Graphic Calculator at https://www.desmos.com/calculator

4. Web based resources
   - Khan academy at: http://www.**khanacademy**.org
     - Exercises and videos on Limits and derivatives: https://www.khanacademy.org/exercisedashboard
   - You tube at: http://www.youtube.com
   - **Google** at: http://www.google.com

5. Calculus I - Practice Problems
   - Paul's Online Math Notes: http://tutorial.math.lamar.edu/problems/calci/calci.aspx

Chapter 3 (Page 115)
Differentiation

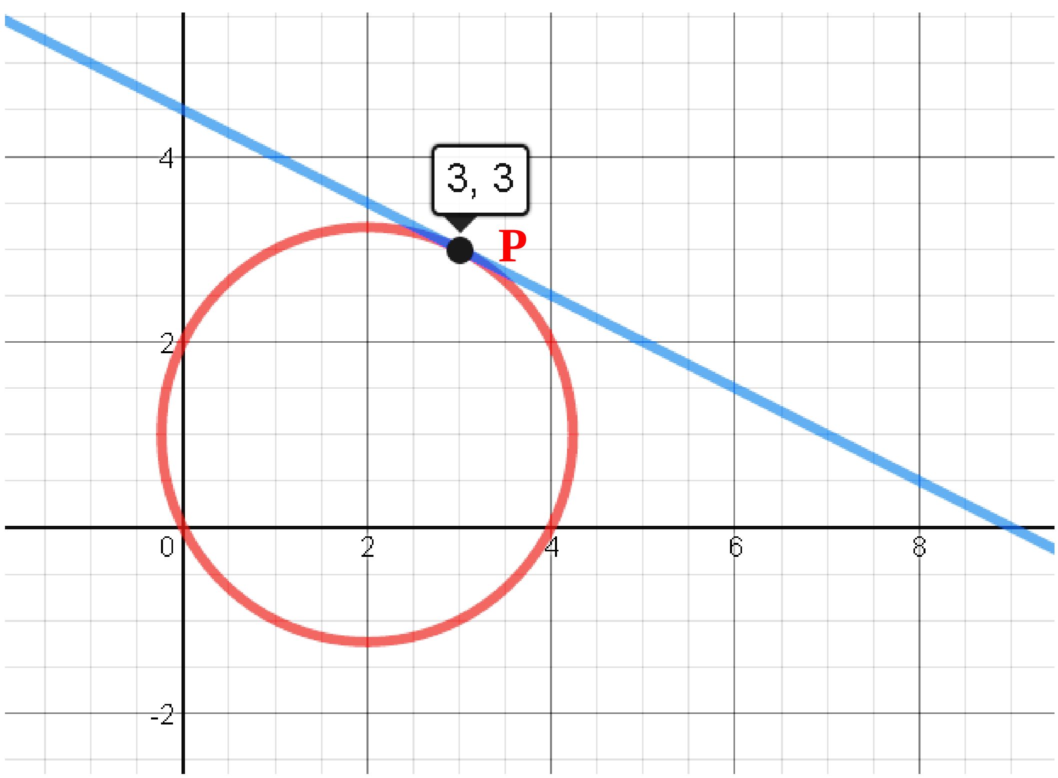
Objectives: By the end of this chapter you should be able to:

- **Find derivative using limit (not easy to do)**
- **Find derivatives using the Rules of Derivatives**
- **Understand Chain Rule and use it for finding derivatives**
- **Find derivatives using Implicit Differentiation**
- **Find derivatives of inverse functions**

Tangent Lines

Calculus is the branch of mathematics that studies rates of change of functions. The rate (or slope) is the same at every point on a line. However, for graphs other than lines, the rate at which the graph rises or falls, changes from point to point.

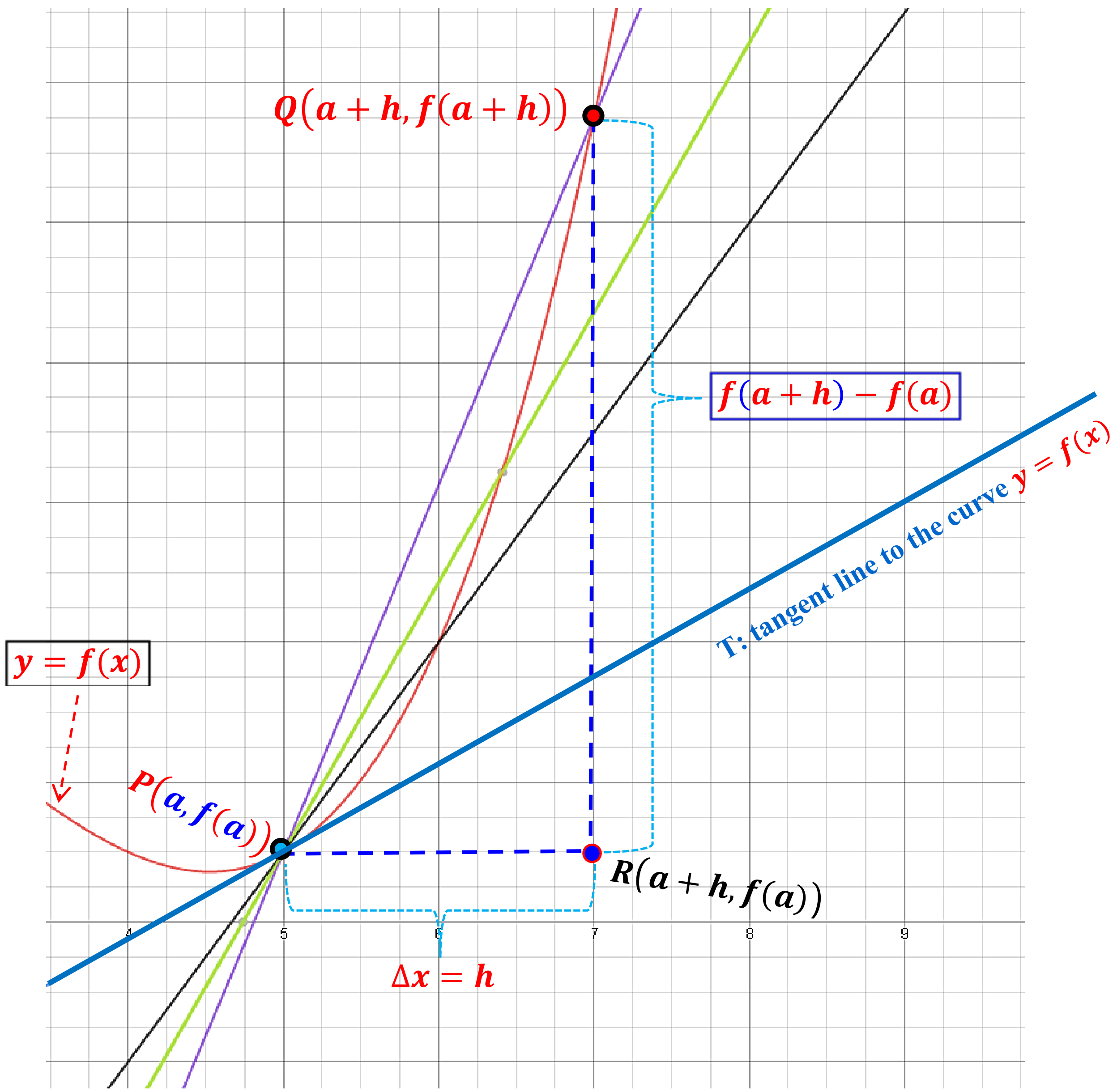
In geometry, a tangent line to a circle is defined as a line that touches the circle at only one point. In the figure below the blue line is tangent to the circle $(x-2)^2+(y-1)^2=5$ at point $\mathbf{P}(3,3)$.



Slope of a tangent line to a curve

Consider the graphs below: we a want to find the slope, hence equation of the tangent line **T** to the curve $y = f(x)$ at point $P(a, f(a))$. To find the slope of **T**, consider the secant line **PQ**

Note that the slope of the secant line: $PQ = \frac{f(a+h) - f(a)}{h}$



Slope of the tangent line T = slope of the secant line as the point **Q** slides to point **P**

= slope of the secant line as $h = \Delta x$ **goes zero**

That means; the **slope of the tangent line** is the limit:

$$\lim_{h \to o} \frac{f(a+h)-f(a)}{h} = \lim_{\Delta x \to 0} \frac{f(a+\Delta x)-f(a)}{\Delta x}$$

This is also defined to be **the slope of the curve** at the point **P.**

Definition of the Derivative

The derivative of the function f **at a point** a is denoted by $f'(a)$ and is defined as:

$f'(a) = \lim_{h \to o} \frac{f(a+h)-f(a)}{h} = \lim_{\Delta x \to 0} \frac{f(a+\Delta x)-f(a)}{\Delta x}$**,** provided this limit exists.

The notation $f'(a)$ *is read as*; "f prime of a" *or* "f prime at a"

Note:

1) The function $f'(x)$ is called the **derivative** of f **with respect to** x **and is given by**.

   $f'(x) = \lim_{h \to o} \frac{f(x+h)-f(x)}{h}$**, provided this limit exists**

2) The process of finding the derivative is called **differentiation**. A function is **differentiable** at x if its **derivative exists** at x.

Notations:

Most common notations used to **denote** the **derivative** of $y = f(x)$ are:

$$f'(x), \quad \frac{dy}{dx}, \quad y', \quad \frac{d}{dx}[f(x)], \quad D_x f(x)$$

Note:

1) $f'(a) = \lim_{h \to o} \frac{f(a+h)-f(a)}{h}$:
   - Is the derivative of the function at the point $x = a$.
   - Is the slope of the tangent line at the point $(a, f(a))$
2) $f'(x) = \lim_{h \to o} \frac{f(x+h)-f(x)}{h}$ is the **derivate** at any point x
3) If f is differentiable at $x = a$ then:
   a) **Derivative** of f at $x = a$ exists
   b) **The graph of** f **has a tangent line** at $x = a$.
4) If f is **not differentiable** at $x = a$, then either the graph of f has no a well-defined tangent at $x = a$, or it has a vertical tangent line at $x = a$, or f has discontinuity at $x = a$

Example 1: Find the **derivative** of the following **functions** at the indicated **points**

a) $\boldsymbol{f(x) = 2x - 3}$**, at** $\boldsymbol{x = 2}$

b) $\boldsymbol{f(x) = x^2}$ **, at** $\boldsymbol{x = -3}$

c) $\boldsymbol{f(x) = \sqrt{x}}$**, at** $\boldsymbol{x = 9}$

Example 2: Find the **slope** and **equation** of the **tangent line** to the curve $\boldsymbol{f(x) = x^2}$ at the point $(\mathbf{2}, \mathbf{4})$.

Example 3: Find the derivative of the following functions using the definition (i.e. using limit).

a) $f(x) = C$

b) $f(x) = x$

c) $f(x) = x^3$

d) $f(x) = \sin x$

e) $f(x) = e^x$

f) $f(x) = 2x - 3$

g) $f(x) = x^2$

h) $f(x) = \sqrt{x}$

i) $f(x) = \cos x$

Example 4: Graphs with a **sharp turns** or **cusps**

a) Let $\boldsymbol{f(x) = |x|}$, show the **derivative** of $\boldsymbol{f}$ at $\boldsymbol{x = 0}$ **does not exist (**sharp turn at 0**)**

b) $\boldsymbol{f(x) = \sqrt[3]{x^2}}$, show the **derivative** of $\boldsymbol{f}$ at $\boldsymbol{x = 0}$ **does not exist (**cusp at 0**)**

c) $\boldsymbol{f(x) = \sqrt[3]{x}}$, show the **derivative** of $\boldsymbol{f}$ at $\boldsymbol{x = 0}$ **does not exist**

Example 5: Sketch the graphs of the functions in Example 4 above and discuss the tangent lines at the given points.

Note: Derivatives does not exist at sharp turns or cusps

Differentiability and Continuity

Theorem 3.1 If $\boldsymbol{f}$ is **differentiable** at $\boldsymbol{x = c}$, then $\boldsymbol{f}$ is **continuous** at $\boldsymbol{x = c}$.

Proof: To show $\boldsymbol{f}$ is continuous at $\boldsymbol{x = c}$, we need to show:

$\lim_{x \to c} f(x) = f(c)$, which is equivalent to $\lim_{x \to c}[f(x) - f(c)] = 0$

$$\lim_{x \to c}[f(x) - f(c)] = \lim_{x \to c}\left[\frac{f(x)-f(c)}{x-c}(x - c)\right]$$

$$= \lim_{x \to c}\left[\frac{f(x)-f(c)}{x-c}\right]\lim_{x \to c}(x - c)$$

$$= f'(c) \times 0$$

$$= 0 \qquad \blacksquare$$

Note If $\boldsymbol{f}$ is **not differentiable** at $x = a$, then either the graph of $\boldsymbol{f}$ has no a well-defined tangent at $x = a$, or it has a vertical tangent line at $x = a$, or $\boldsymbol{f}$ has discontinuity at $x = a$.

Example 6: Show that the derivative of the function $\boldsymbol{f(x) = \begin{cases} \frac{|x^2|}{x} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}}$ does not exist **at** $\boldsymbol{x = 0}$

Example 7: Is the function $f(x) = \sqrt{x - 2}$ differentiable at $x = 3$? At $x = 1$?

Wiki-Book Exercise: https://en.wikibooks.org/wiki/Calculus/Differentiation/Basics_of_Differentiation/Exercises
Question #1 – 10
Practice Problems Larson & Edwards, **5th** Editions

Page 123 Exercise 3.1, 5, 9, 17, 21, 23, 25, 31, 33, 38, 43, 83, 84, 85

3.2 Basic Differentiation Rules and Rates of Change (Page 127)

Objectives: By the end of this section you should be able to

- Find the derivative of a function using the constant- multiple rule
- Find the derivative of a function using Power Rule
- Find the derivative of a function using sum and difference rules
- Find the derivative of a function using product and quotient rules
- Use derivative to find rate of change

Theorem (Constant Rule)

If $f(x) = k$, where k is any real number, then $f'(x) = 0$ (i.e.the **derivative** of a **constant** is **0**.)

Proof: Use limit

Example 1: $f(x) = 34$

Theorem (Constant Multiple Rule)

If f is a differentiable function and c is a real number, then cf is also differentiable and

$$\frac{d}{dx}[cf(x)] = c\frac{d}{dx}f(x) = cf'(x)\,.$$

Proof: Use limit

Example 2: $f(x) = -12x^2$

Theorem (The Power Rules)

If **n** is a real number, then the function $f(x) = x^n$ is differentiable and

$$\frac{d}{dx}[x^n] = nx^{n-1}$$ **Power Rule**

For f to be differentiable at **x = 0, n** must be a number such that x^{n-1} is defined on an interval containing **0**

Proof: Use limit

Derivatives Using Rules (Power Rule)

Example 3: $f(x) = x^3$

Example 4: $f(x) = x = x^1$

Example 5: $f(x) = \sqrt{x} = x^{1/2}$

Theorem (The Sum Difference Rules)

The **sum or difference** of two **differentiable** functions $\boldsymbol{f}$ and $\boldsymbol{g}$ is itself **differentiable**.

Moreover: $\frac{d}{dx}[f(x)+g(x)] = f'(x)+g'(x)$ **Sum Rule**

$\frac{d}{dx}[f(x)-g(x)] = f'(x)-g'(x)$ **Difference Rule**

Proof: Use limit

Derivatives using derivative rules

Example 6: Find the derivative for each of the following:

a) $f(x) = 4x^3 - x^2 + 3x + 2$

b) $f(x) = 6x^3 + 3x^2 - 5x + 1$

c) $f(x) = x^{-4} + e^x + \pi x$

d) $y = 4\sqrt{x} + \sqrt[3]{x} - e$

e) $f(x) = x^{-5} - x^{-2} + 11x^{-1}$

f) $f(x) = \frac{6}{x} - \frac{2}{\sqrt{x}}$

g) $y = \frac{x^2+2x-1}{x}$

h) $y = \frac{x^2+5x-3}{\sqrt{x}}$

Example 7: Find $f'(9)$ for $f(x) = \frac{3}{x} + 2\sqrt{x}$.

Example 8: Find the **slope** of the **tangent** line to the graph of $f(x) = x^3 - 4x + 2$ at $x = 1$

Example 9: Find the values of x where the tangent line is **horizontal** for $f(x) = x^3 - 2x^2$.

Example 10: Find the **equation** of the **tangent** line to the curve for $f(x) = 2x^2 - 1$ at $x = 3$.

Example 11: Find the derivative of the following functions

a) $f(x) = e^x - 2\sin x + \cos x + x^2 - 3x - 34$

b) $y = -2e^x + \pi\cos x - 4x^{\frac{3}{2}} + e$

Wiki-Book Exercise: https://en.wikibooks.org/wiki/Calculus/Differentiation/Basics_of_Differentiation/Exercises
Question # 11 – 19

Practice Problems: Larson & Edwards, 5th Editions
Page 136, exercises 3.2: 5, 9, 17, 25, 27, 29, 37, 39, 45, 49, 53, 58, 60, 61, 67, 69, 81, 83, 91 – 96, 103, 111, 112, 117, 121, 123

3.3 Product and Quotient Rules and Higher - Order Derivatives (Page 140)

Objectives: By the end of this section you should be able to:

- Understand and state the Product and Quotient Rules
- Find the derivatives of functions using the Product and Quotient Rules
- Find products of trigonometric functions
- Find higher – order derivatives

The Product and Quotient Rules

Theorem 3.8/3.9

Let $\boldsymbol{f}$ *and* $\boldsymbol{g}$ be **differentiable** functions. Then $\boldsymbol{f} \times \boldsymbol{g}$ and $\boldsymbol{f}/\boldsymbol{g}$ are **differentiable** functions and

a) $\frac{d}{dx}[f(x)g(x)] = f'(x)\,g(x) + f(x)\,g'(x)$ **(Product Rule)**

b) $\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] = \frac{f'(x)\,g(x) - f(x)\,g'(x)}{[\,g(x)\,]^2}$, **where** $g(x) \neq 0$ **(Quotient Rule)**

Proof: Follows from limit Definition of derivatives

Example 1: Find the derivatives of the following functions

a) $h(x) = (5x - 4)(2x + 11)$

b) $p(x) = (5x^2 - 4x)(2x^3 + 11)$

c) $h(x) = 3\cos x\,(2 - \sin x)$

d) $h(x) = \frac{x}{2x+5}$

e) $r(x) = \frac{x^2-3x}{2x+5}$

f) $r(x) = \frac{1-\cos x}{\sin x}$

Example 2: Find the derivatives of the following functions

a) $l(x) = \tan x$

b) $h(x) = \csc x$

c) $l(x) = \sec x$

d) $f(x) = \cot x$

Example 3: Find the **equation** of **the tangent line** to the graph of $f(x) = (2x^2 + 3x + 1)(x + 5)$ at $\boldsymbol{x = 0}$

Example 4: Find the **equation** of the **tangent line** to the graph of the function $f(x) = \frac{x^2+x}{x-1}$ at $x = 2$

Example 5: Find the derivative

a) $f(x) = \frac{\sqrt{x}}{x+1}$

b) $g(x) = \frac{x^4-16}{x^4+17}$

c) $h(w) = (w + 1)(3w^2 + 1)$

Wiki-Book Exercise: https://en.wikibooks.org/wiki/Calculus/Differentiation/Basics_of_Differentiation/Exercises
Question # 20 – 30

Rates of Change

Rates of Change two types:

- **Average rate of change**
- **Instantaneous rate of change**

1) **The Average rate of change** of $y = f(x)$ with respect to x for a function f as x changes from a to b is given by: $\frac{f(b)-f(a)}{b-a}$

Example: Find the average rate of change for the following.

a) $y = x^2 + 2x$ between $x = 0$ and $x = 3$ b) $f(x) = \sqrt{x}$ between $x = 1$ and $x = 4$

2) **The Instantaneous Rate of change:** Instantaneous rate of change for a function f at a point a is defined by the following limit $\lim_{h \to o} \frac{f(a+h)-f(a)}{h}$

Examples of **Instantaneous** rate of changes are

- **Velocity** (The Speedo Meter reading of a car)
- **Accelerations**
- **Marginal Profit**
- **Marginal Revenue**
- **Marginal cost**

Example 1: Find the **instantaneous** rate of change for

a) $y = x^2 + 2x$ at x = 0 and at x = 3 b) $f(x) = \sqrt{x}$ at $x = 1$ and at $x = 4$

Example 2: For exercise 1) and 2) below use the position function $S(t) = -16\,t^2 + V_0 t + S_0$ be the for free-falling objects.

1) A silver dollar is dropped from the top of a building that is 1362 foot tall
   a) Determine the position and velocity function for the coin
   b) Determine the average velocity on the interval [1, 2]
   c) Find the instantaneous velocity when t =1 and t = 2
   d) Find the time required for the coin to reach ground level
   e) Find the velocity of the coin at impact
2) A ball is thrown straight down from the top of a 220 foot building with an initial velocity of – 22 feet per sec. What is the velocity after 3 sec? What is its velocity after falling 108 foot?

Example 3: Suppose that the total profit in hundreds of dollars from selling x items is given by: $p(x) = 2x^2 - 5x + 6$. Find the marginal profit at $x = 2$.

Example 4: A ball is thrown vertically upward so the height in feet of the ball above the ground t seconds after its release is given by the function $s(t) = -16t^2 + 29t + 6,\ 0 \le t \le 2$

a) Find the instantaneous velocity of the ball at $t = 0.5\ s$
b) Find the velocity $v(t) = s'(t)$ for $0 < t < 2$. What is the velocity of the ball just before impacting the ground?

Higher- Order Derivatives

Higher derivatives and their notations:

$f'(x)$ --- 1st derivative (gives us slope, tangent line as well as intervals for increasing and decreasing)

$f''(x)$ -- 2nd derivative (gives us intervals of concavity)

$f'''(x)$ -- 3rd derivative

$f^{(4)}(x)$ -- 4th derivative (Note from 4th order derivative up wards no "Prime" notation)

etc.

$f^{(n)}(x)$ -- nth derivative

Example 1: Find the indicated derivatives for $f(x) = 3x^4 - x^3 + 5x^2 - 4$

a) $f'(x)$
b) $f''(x)$
c) $f^{(3)}(x)$
d) $f^{(4)}(x)$
e) $f^{(5)}(x)$

Example 2: We can also evaluate higher order derivatives at a point. **For example**, find$f''(2)$ for

$f(x) = 4x^3 - 5x^2 + 11$

Example 3: Find $f''(2)$ for each of the following.

a) $f(x) = 4x^3 + 2x^2 + 3x$
b) $f(x) = 5e^x(2x^3 - 3x)$
c) $f(x) = 4x \sin x$

Example 4: The velocity of an object in meter per second is $(t) = 36 - t^2$, $0 \le t \le 6$. Find the velocity and acceleration of the object when t = 3.

Example 5: Develop a rule for $f^{(n)}(x)$ given $f(x)$.

a) $f(x) = x^n$
b) $f(x) = \frac{1}{x-1}$

Example 6: Let $n > 0$ find the **second** derivatives of the following functions

a) $f(x) = x^n \sin x,$

b) $f(x) = \frac{\cos x}{x^n}$

Paul's Online Math Notes &
Exercises: http://tutorial.math.lamar.edu/Problems/CalcI/HigherOrderDerivatives.aspx

Practice Problems Larson & Edwards, **5th** Editions
3.3 Exercises, Page 147, 1, 3, 5, 7, 11, 25, 27, 33, 37, 39, 43, 47, 51, 53, 55, 75, 77, 131 – 136, 137, 139

3.4 The Chain Rule (Page 151)

Objectives: By the end of this section you should be able to

- State the chain rule
- Find derivatives of composite functions
- Simplify derivatives and find derivatives of function using the general power rule
- Find the derivatives of transcendental function using the Chain Rule
- Define and differentiate exponential functions that have bases other than e

Recall:

The **composite function** or the composition of f and g, $f \circ g$ is defined as $(f \circ g)(x) = f(g(x))$, where x is in the **domain** of g and $g(x)$ is in the **domain** of f.

Example 1: Given $f(x) = x^2 - 3$ and $g(x) = 2x + 1$, find

a) $(f \circ g)(x)$
b) $(f \circ g)(1)$
c) $(g \circ f)(x)$
d) $(g \circ f)(-2)$

Sometimes we know the composite and would like to know the original functions.

Example 2: Find $f(x)$ and $g(x)$ such that $h(x) = (f \circ g)(x) = f(g(x))$.

a) Decompose the function $h(x) = (4 + 3x)^5$

b) Decompose the function $p(x) = \sqrt{x^2 + 4}$

c) Decompose the function $r(x) = e^{5x-3}$

In this section we would be able to find the derivatives of composite functions.

Example 3: We can easily find the derivative of $y = (2x + 3)^2$ by the Product Rule. If the question were, find the derivative of $y = (2x + 3)^7$ or $f(x) = \sqrt[3]{(x^2 - 3x + 11)^4}$, our old way would be very tedious and messy. We need a new way.

Theorem 3.11 (The Chain Rule, Derivative of composite functions**):**

Let $y = f(g(x))$, then $y' = f'(g(x)) * g'(x)$. (We are taking the derivative of the "**outside**" **function** first and multiply this with the **derivative** of the **inside** function)

Proof: Refer to the book

Example 1: Using the Chain Rule find the derivatives

a) $y = (x^3 + 1)^3$ b) $y = (2x + 3)^7$ c) $f(x) = \sqrt[3]{(x^2 - 3x + 11)^4}$

Theorem 3.12: The General Power Rule

If $y = [u(x)]^n$, where u is a differentiable function of x, and n is any real number, then $\frac{dy}{dx} = n[u(x)]^{n-1}\frac{du}{dx}$ or equivalently $\frac{d}{dx}[u^n] = n\,u^{n-1}u'$.

Proof: Follows from the Chain Rule

Example 2:

a) Find the derivative of $f(x) = \sqrt[3]{(x^2 - 3x + 11)^4}$

b) Find all points on the graph of $f(x) = \sqrt[3]{(x^2-1)^2}$ for which $f'(x) = 0$ and those for which $f'(x)$ does not exist.

Example 3: Do Example 6, 7, 8, and 9 pages 154 and 155.

Example 4: Find the derivative.

a) $y = x^2(5x+2)^3$

b) $y = x^2\sqrt{4x+1}$

c) $y = \frac{x}{\sqrt{2x-3}}$

d) $y = (3x-1)^2(x+7)^{-3}$

Example 5: Find the **equation** of the **tangent line** to the graph of the function $f(x) = (x^3+7)^{2/3}$ at $x = 1$.

Wiki-Book Exercise: https://en.wikibooks.org/wiki/Calculus/Differentiation/Basics_of_Differentiation/Exercises
Question # 31 – 41

Transcendental Functions and Chain Rule

Note: By Chain Rule $\frac{d}{dx}\left[e^{f(x)}\right] = e^{f(x)}f'(x)$

Example 6: Find the derivative for each

a) $f(x) = e^{x^2}$

b) $y = 2e^x$

c) $y = e^{x^2+2x}$

d) $y = e^{(3x+2)^2}$

e) $f(x) = e^2$

f) $y = 2x^3e$

g) $f(x) = e^{x^3} + x$

Example 7: Find the derivative for each.

a) $f(x) = xe^x$

b) $y = x^3e^{-2x}$

c) $y = (4x-3)e^{2x}$

d) $y = \frac{e^x}{2x+1}$

e) $f(x) = (x+5)^3e^{-2x^2}$

Example 8: Find the derivative of the following functions

a) $y = \sin(x^2+2)$

d) $y = \tan(x^2-2)$

b) $y = \sin^3(x^2+2)$

e) $y = \cos(3x^2+2x)$

c) $y = \cos\sqrt{x^2+1}$

Example 9: Reading Example 10, Example 11, and Example 12 page 156

The Derivative of the Natural Logarithm Function

Theorem 3:13 Derivative of the **Natural Logarithm** Function

Let u be a differentiable function of x

1) $\frac{d}{dx}[\ln x] = \frac{1}{x}$, $x > 0$
2) $\frac{d}{dx}[\ln u] = \frac{1}{u}\frac{du}{dx}$, $u > 0$. **Here u is a function of x**

Proof: of 1) Note $y = \ln x$ is equivalent to say $e^y = x$. Thus,

$$\frac{d}{dx}[e^y] = \frac{d}{dx}[x]$$

$$e^y\frac{d}{dx}[y] = 1 \qquad \textbf{why?}$$

$$\frac{d}{dx}[y] = \frac{1}{e^y} = \frac{1}{x}$$

Example 1: Find the derivative for each of the following

a) $y = \ln x^2$ b) $y = \ln(3x^2 + 4x)$ c) $y = \ln(5x^2)$

d) $y = 6\ln x^2$ e) $y = \ln\sqrt{2x+5}$ f) $y = \ln(2x+3)^3$

Example 2: Read Examples on pages 157 and 158

Theorem 3:14 Derivative involving absolute value

Let u be a differentiable function of x such that $u \neq 0$, then $\frac{d}{dx}[\ln(u)] = \frac{u'}{u}$.

Proof:

Theorem 3:14 Derivative for Bases Other than e

Let $a > 0$ and $a \neq 1$ and Let u be a differentiable function of x. **Then**

1) $\frac{d}{dx}[a^x] = (\ln a)a^x$
2) $\frac{d}{dx}[a^u] = (\ln a)a^u\frac{du}{dx}$
3) $\frac{d}{dx}[\log_a x] = \frac{1}{(\ln a)x}$
4) $\frac{d}{dx}[\log_a u] = \frac{1}{(\ln a)u}\frac{du}{dx}$

Example 3: Find the derivative for each.

a) $y = e^{x^2}\ln x$ b) $y = \frac{5x^2}{\ln x}$ g) $y = x^{\sin x}$

c) $y = (x^2+3)\ln(x+4)$ d) $y = \frac{\ln x}{e^x}$ h) $y = x^{x^x}$

e) $y = \ln[(x+4)(x^2+3)]$ `f) $y = x^x$ i) $y = x^{\cos x}$

Example 4: Examples on page 160 of the Book

Wiki-Book Exercise: https://en.wikibooks.org/wiki/Calculus/Differentiation/Basics_of_Differentiation/Exercises
Question # 42 – 61

Practice Problems Larson & Edwards, **5th** Editions
Page 161, Exercises 3.4, #1 – 8, 11, 15, 23, 27, 35, 37, 57, 61, 67, 69, 77, 79, 85, 91, 97, 101, 103, 107, 111, 115, 121, 133, 135, 168, 173

3.5 Implicit Differentiation (page 166)

Objectives:

- Distinguish between functions written in implicit form and explicit form
- Use implicit differentiation to find derivatives of functions
- Find derivatives of functions using logarithmic differentiation

Implicit and Explicit Functions

Explicit Equations or Equations given in Explicit Form	Implicit Equations or Equations given in Implicit Form
$y = 3x + 9$, **y is expressed explicitly as a function of x**	$yx = 1$, **y is not expressed explicitly as a function of x**
$y = x^3 - 12x^2 + 4x - 9$	$x^2 + y^2 = 16$
$y = -3x^3 - 4x + 2$	$x^2 - 2y^2 + 4y = 8$

Implicit Differentiation:

The **process** of **finding** $\frac{dy}{dx}$ from an equation defining **y implicitly** as a **function** of x is called **Implicit Differentiation.**

Example 1: Find dy/dx by implicit differentiation.

a) $2x^2 - y^2 = 4$ c) $\sin y = e^{3x}$ e) $y^3 + y^2 - 5y - x^2 = -4$

b) $8xy = 10$ d) $x^2e^y + y = x^3$ f) $e^x \ln y = y$

Example 2: Find the equation of the line tangent to the curve at the given point.

a) $x^2 + y^2 = 25; (-3, 4)$ b) $2y^2 - \sqrt{x} = 4\,; (16,\ 2)$

Example 3: The position of a particle at time t is given by s, where $s^3 + 2st + 4t^3 - 8t = 0$.
Find the velocity ds/dt.

Example 4: Use implicit differentiation to find an equation of a tangent line to the hyperbola
$\frac{x^2}{6} - \frac{y^2}{8} = 1$ at (3, -2).

Example 5: Find $\frac{d^2y}{dx^2}$ in terms of x and y

a) $x^2 + y^2 = 4$ b) $x^2y^2 - 2x = 3$ c) $x^2 - y^2 = 25$

Example 6: Page 167 – 170, Do Examples 2, 4, 5, 7, 8, and 9

Example 7: Find dy/dx using Logarithmic differentiation

a) $y = \frac{x^2\sqrt{3x-2}}{(x+1)^2}, x > \frac{2}{3}$ b) $y = x^{\ln x}, x > 0$ c) $y = x^{\tan x}$

Wiki-Book Exercise: https://en.wikibooks.org/wiki/Calculus/Differentiation/Basics_of_Differentiation/Exercises
Question # 62 – 76
Practice Problems Larson & Edwards, **5th** Editions
Exercise 3.5, Page 171: 1 – 4, 11, 13, 15, 17, 19, 31, 37, 43, 59, 61, 71, 73, 75

Related Rates

Objectives: By the end of this section you should be able to

- Find related rates
- Use related rates to solve real life problems

Related rates problems involve finding a rate at which a quantity changes by relating that quantity to other quantities whose rates of change are known. The rate of change is usually with respect to time.

Wiki – Book Examples: https://en.wikibooks.org/wiki/Calculus/Related_Rates#Examples

Example 1: Suppose x and y are differentiable functions of t and are related by the equations given below. Evaluate dy/dt at the given point.

a) $y = x^2 + 3; \frac{dx}{dt} = 2,\ x = 1$
b) $y^2 - 5x^2 = -1\ ;\ dx/dt\ =\ -3, y\ =\ 2,\ x = 1$
c) $xe^y = 2 - \ln 2 + \ln x\ ;\ dx/dt = 6, y = 0, x = 2$

Example 2: A rock is thrown into a still pond. The circular ripples move outward from the point of impact of the rock so that the radius of the circle formed by a ripple increases at the rate of 2 feet per minute. Find the rate at which the area is changing at the instant the radius is 6 feet.

Example 3: A ladder 25 feet long rests against a vertical wall. If the bottom of the ladder slides away from the wall at a rate of 2 feet/sec, how fast is the top of the ladder sliding down the wall when the bottom of the ladder is 7 feet from the wall?

Example 4: One airplane is approaching an airport from the north at 450 mph. A second airplane approaches from the east at 600 mph. Find the rate at which the distance between the planes changes when the southbound plane is 150 miles away from the airport and the westbound plane is 200 miles from the airport.

Example 5: The formula for the volume of a cone is $V = \frac{1}{3}\pi r^2 h$. Find the rate of change of the volume if dr/dt is 2 inches per minute and $h = 3r$ when a) r = 6 in and b) r = 24 in.

Example 6: A man 6 feet tall walks at a rate of 5 feet per sec away from a light that is 15 feet above the ground. When he is 10 feet from the base of the light:

a) At what rate is the tip of his shadow moving?
b) At what rate is the length of his shadow is changing?

Repeat **Example 6**, for a man 6 feet tall walking at a rate of 5 feet per sec toward a light that is 20 feet above the ground

Example 7: Reading, Examples 1, 2, 3, 4, 5, 6 Page 182 – 186

Wiki –Book: https://en.wikibooks.org/wiki/Calculus/Related_Rates#Exercises **Exercises #1 – 4**

Paul's Online Notes & Exercises: http://tutorial.math.lamar.edu/Problems/CalcI/RelatedRates.aspx

Practice Problems Larson & Edwards, **5th** Editions

Page187: 1, 3, 5, 7, 11, 13, 15, 21, 23, 25, 27, 29, 31, 35, 41,

Derivatives of Inverse Functions (Page 175)

Objectives: By the end of this section you should be able to

- Find the derivative of an inverse function
- Differentiate an inverse trigonometric function
- State the Basic Differentiation Rules for Elementary functions

Inverse Functions

Definition: (The Inverse of a Function)

The inverse of a function $\boldsymbol{f}$ is a **relation defined** as the **set**

$$\{(y,\ x): \textbf{whenever } (x, y) \textbf{ belongs to } f\}$$

Example 1: **Let** $\boldsymbol{f}$ be the function **given below**

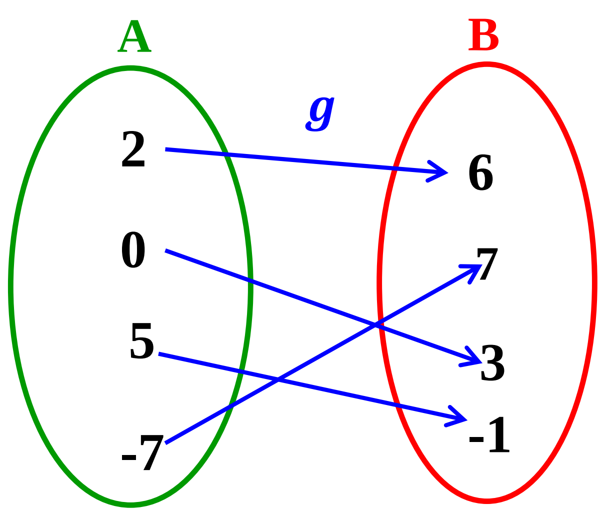
$$f = \{(2,\ 6), (0, 3), (5, 3), (-7, 7)\}$$

Inverse of $f = \{(6, 2), (3, 0), (3, 5), (7, -7)\}$

The **inverse** of $\boldsymbol{f}$ **is not** a function. **Why?**

On the other hand consider the function in **Example 2**

Example 2: Consider the function $g : A \rightarrow B$**,** means that $\boldsymbol{g}$ is a function from **A into B.**



$g = \{(2, 6), (0, -5), (5, 3), (-7, 7)\}$

$g^{-1} = \{(6,\ 2), (-5,\ 0), (3,\ 5), (7, -7)\}$

The inverse of $\boldsymbol{g}$ **is a function**

Note: The inverse of a function $\boldsymbol{f}$ is a **function** if the **original function** is **one – to – one**

Notation: The inverse of a **one – to – one function** $\boldsymbol{f}$ **,** is denoted by f^{-1}**.**

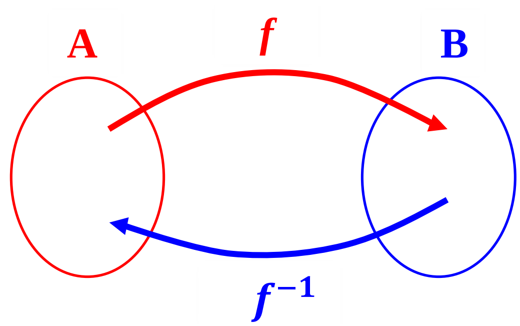
f^{-1} is a **function** and is given by: $f^{-1} = \{(y,\ x): whenever\ (x, y)\ belongs\ to\ f\}$

Inverse function Property

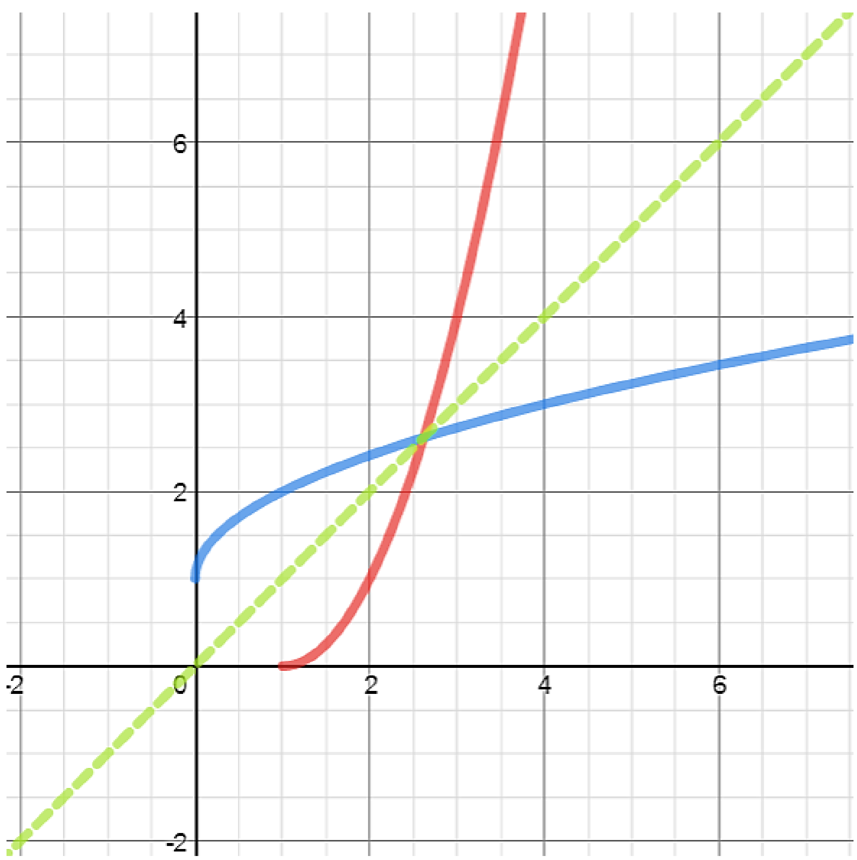
Let f be a **one – to – one function** with **domain A** and **Range B**. The inverse function f^{-1} has **domain B** and **range A** and satisfies the following **cancellation properties**:

- $f^{-1}(f(x)) = x$ for every x in **A**
- $f\left(f^{-1}(x)\right) = x$ for every x in **B**

Conversely any function f^{-1} **satisfying these equations** is the **inverse** of f



Example 3: Graph of $f(x) = (x-1)^2, x \geq 1$ and its inverse $f^{-1}(x) = \sqrt{x} + 1$



Derivative of Inverse Functions

Theorem 3.16: Continuity and Differentiability of Inverse Functions

Let $\boldsymbol{f}$ be a function whose domain is an interval I. If $\boldsymbol{f}$ has an inverse function then the following statements are **TRUE**:

a) If $\boldsymbol{f}$ is continuous in its domain then $\boldsymbol{f^{-1}}$ is continuous in its domain

b) If $\boldsymbol{f}$ is differentiable on an interval containing c and $\boldsymbol{f'(c) \neq 0}$, then $\boldsymbol{f^{-1}}$ is differentiable at $\boldsymbol{f(c)}$.

Proof: Refer to the book

Theorem 3.17: The derivative of an Inverse Function

Let $\boldsymbol{f}$ be a function that is differentiable on an interval $\boldsymbol{I}$. If $\boldsymbol{f}$ has an inverse function $\boldsymbol{g}$, then $\boldsymbol{g}$ is differentiable at any $\boldsymbol{x}$ for which $\boldsymbol{f'\big(g(x)\big) \neq 0}$. Moreover,

$$\boldsymbol{g'(x) = \frac{1}{f'\big(g(x)\big)}, \quad f'\big(g(x)\big) \neq 0}$$

Proof: Refer to the book

Example 4: Evaluating the derivatives of inverse functions

Let $f(x) = \frac{1}{4}x^3 + x - 1$

a) What is the value of $f^{-1}(x)$ when x = 3?

b) What is the value of $(f^{-1})'(x)$when x = 3?

Solution: $\boldsymbol{f}$ is invertible. Why?

Is $\boldsymbol{f}$ one – to – one?

a) We want to find $f^{-1}(3)$

Let $f^{-1}(3) = t$, this implies $f(t) = 3$.

That is $f(t) = \frac{1}{4}t^3 + t - 1 = 3$ which gives t = 2.

Thus, $f^{-1}(3) = 2$

b) We want to find $(f^{-1})'(3)$

By **Theorem 3.17**,

$$(f^{-1})'(3) = \frac{1}{f'\big(f^{-1}(3)\big)} = \frac{1}{f'(2)} = \frac{1}{\frac{3}{4}2^2+1} = \frac{1}{4}$$

Example 5: Graphs of inverse functions have reciprocal slopes

Let $f(x) = x^3$, then $f^{-1}(x) = \sqrt[3]{x}$ is the inverse of $\boldsymbol{f}$. Show that the **slopes** of $f\ and\ f^{-1}$ are **reciprocals** at each of the following points.

a) $(2,\ 8)\ and\ (8,\ 2)$

b) $(-3, -27)\ and\ (-27, -3)$

Solution:

a) At the points $(2,\ 8)\ and\ (8,\ 2)$

$f(x) = x^3$ and $f'(x) = 3x^2$ and so $f'(2) = 3(2^2) = 12$

$f^{-1}(x) = \sqrt[3]{x} = x^{1/3}$ and $(f^{-1})'(x) = \frac{1}{3}x^{-2/3}$ and so

$(f^{-1})'(8) = \frac{1}{3}(8)^{-2/3} = 1/12$

b) Done similarly

Derivatives of Inverse Trigonometric Functions (page 177)

Theorem 3.17b: Derivatives of Inverse Trigonometric Functions

The following holds true whenever the derivative exist:

a) $\frac{d}{dx}[arcsin\ \boldsymbol{x}] = \frac{1}{\sqrt{1-x^2}}$

b) $\frac{d}{dx}[arctan\ \boldsymbol{x}] = \frac{1}{1+x^2}$

c) $\frac{d}{dx}[arcsec\ \boldsymbol{x}] = \frac{1}{|x|\sqrt{x^2-1}}$

d) $\frac{d}{dx}[arccos\ \boldsymbol{x}] = \frac{-1}{\sqrt{1-x^2}}$

e) $\frac{d}{dx}[arccot\ \boldsymbol{x}] = \frac{-1}{1+x^2}$

f) $\frac{d}{dx}[arccsc\ \boldsymbol{x}] = \frac{-1}{|x|\sqrt{x^2-1}}$

Proof:

a) We want to show $\frac{d}{dx}[arcsin\ \boldsymbol{x}] = \frac{1}{\sqrt{1-x^2}}$ where $\boldsymbol{x} \in (-\mathbf{1}, \mathbf{1})$

First of all we have to restrict the domain of the sine function $\boldsymbol{f(x) = \sin x}$ on the interval $(-\boldsymbol{\pi},\ \boldsymbol{\pi})$**;** on this interval $\boldsymbol{f(x) = \sin x}$ is **one – to – one** hence invertible.

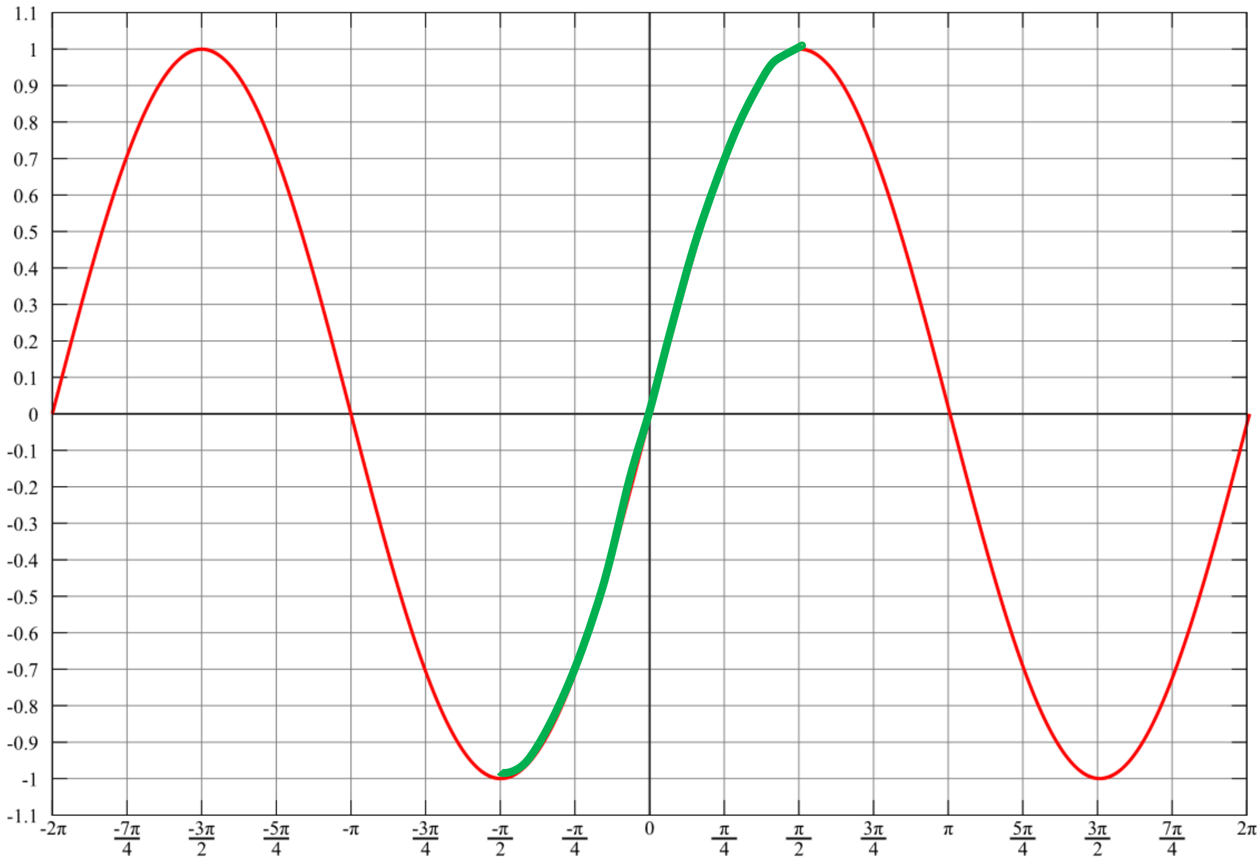
Look at the graph; **green curve**

In other words:

$\boldsymbol{f}: (-\boldsymbol{\pi}, \boldsymbol{\pi}) \to (-\mathbf{1}, \mathbf{1})$**,** is given by $\boldsymbol{f(x) = \sin x}$ is invertible. The inverse sine function

$\boldsymbol{f^{-1}}: (-\mathbf{1}, \mathbf{1}) \to (-\boldsymbol{\pi}, \boldsymbol{\pi})$, is given by $f^{-1}(x) = \sin^{-1} x = \arcsin x$

$y = \sin x$



By **Theorem 3.16** since $f(x) = \sin x$ is differentiable and $f'(c) = \cos c \neq 0 \; \forall \, c \in (-\pi, \; \pi)$ $f^{-1}(x) = \arcsin x$ is differentiable at $f(c)$.

Now let $y = \arcsin x \implies x = \sin y,$ take derivatives on both sides of the last equation:

$\frac{dx}{dx} = \frac{d}{dx}[\sin y]$ which implies $1 = \cos y \; \frac{dy}{dx}$, from this we get

$\frac{dy}{dx} = \frac{1}{\cos y}$, but $y = \arcsin x$, $\cos y = \sqrt{1 - \sin^2 y}$ and $x = \sin y$

So, $\frac{dy}{dx} = \frac{1}{\sqrt{1-\sin^2 y}} = \frac{1}{\sqrt{1 - x^2}}$

Proving $\frac{d}{dx}[\arcsin x] = \frac{1}{\sqrt{1 - x^2}}$

In a similar way we prove b) – f)

Theorem 3.18: Derivatives of Inverse Trigonometric Functions

Let **u** be a differentiable function of $\boldsymbol{x}$.

a) $\frac{d}{dx}[\arcsin \boldsymbol{u}] = \frac{u'}{\sqrt{1-u^2}}$

d) $\frac{d}{dx}[\arccos \boldsymbol{u}] = \frac{-u'}{\sqrt{1-u^2}}$

b) $\frac{d}{dx}[\arctan \boldsymbol{u}] = \frac{u'}{1+u^2}$

e) $\frac{d}{dx}[\operatorname{arccot} \boldsymbol{u}] = \frac{-u'}{1+u^2}$

c) $\frac{d}{dx}[\operatorname{arcsec} \boldsymbol{u}] = \frac{u'}{|u|\sqrt{u^2-1}}$

f) $\frac{d}{dx}[\operatorname{arccsc} \boldsymbol{u}] = \frac{-u'}{|u|\sqrt{u^2-1}}$

Proof: Falls from **The Chain Rule** and **Theorem 2.17b.**

Example 1: Find the derivatives:

a) $y = \arcsin x + x\sqrt{1-x^2}$

b) $y = \ln(t^2+4) - \frac{1}{2}\arctan\frac{t}{2}$

c) $h(t) = \sin(\arccos t)$

d) $y = \frac{1}{2}\left[x\sqrt{4-x^2} + 4\arcsin\frac{x}{2}\right]$

Example 2: Angular Rate of Change

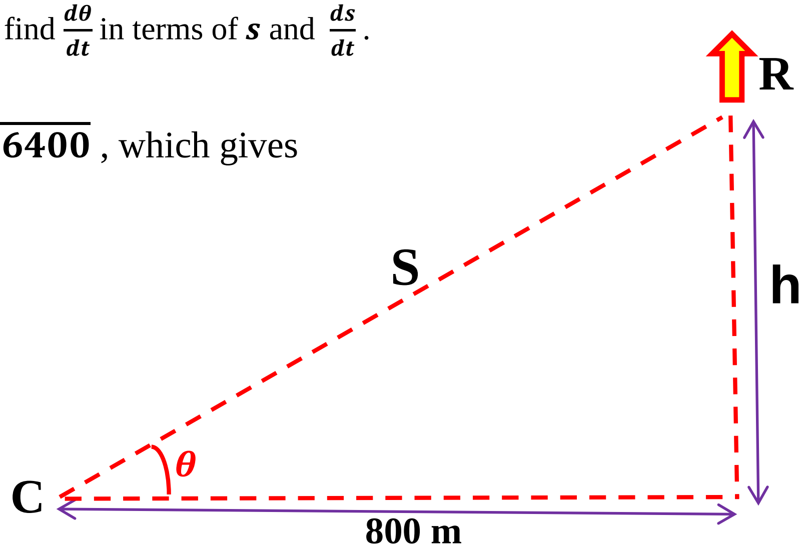
A television camera at ground level is filming the lift-off of a space shuttle at a point 800 meters from the launch pad. Let θ be the angle of elevation of the shuttle and let **s** be the distance between the camera and the shuttle. Write θ as a function of s for the period of time when the shuttle is moving vertically. Differentiate the result to find $\frac{d\theta}{dt}$ in terms of s and $\frac{ds}{dt}$.

Solution:

$\sin\theta = \frac{h}{s}$, where $h = \sqrt{s^2-6400}$, which gives

$\theta = \arcsin\frac{\sqrt{s^2-6400}}{s}$

$\frac{ds}{dt} = \frac{d}{dt}\left[\arcsin\frac{\sqrt{s^2-6400}}{s}\right]$



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